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Properties of Analytic Splines (I) Complex Polynomial Splines

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INTRODUCTION

In a recently published paper [1], the concept of complex cubic spline on a rectifiable Jordan curve was discussed together with the corresponding extension to the associated analytic spline. We introduce here complex polynomial splines which represent, simultaneously, extensions of complex cubic splines and of periodic (real) polynomial splines [2, 3].

This generalization, together with the modifications in the methods of analysis which it necessitates, leads to new properties and serves to shed more light upon the structure of the spline itself. We discuss complex polynomial splines with some of their elementary properties, the associated analytic splines in various representations. We then examine in particular multiple interpolation at a point by analytic splines. In this regard, we both develop and extend some previously announced results [4].

DEFINITIONS AND ELEMENTARY PROPERTIES. Let Γ be a Jordan curve in the complex plane with interior R, and let the points $t_1, t_2, \ldots, t_N$ be located on Γ in counter-clockwise order, forming a mesh Δ. Denote by Γ_j the arc

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of Γ extending from t_{j-1} to t_j (t_0 is to be identified with t_N). By a complex polynomial spline of degree n on Γ associated with Δ (i.e., having knots at t_j, $j = 1, 2, \ldots, N$) is meant a function $q_\Delta(t)$ in $C^{n-1}(\Gamma)$ such that $q_\Delta(t)$ coincides on each Γ_j with a polynomial of degree n.

Denote by $q_j(t)$ the polynomial on Γ_j with which $q_\Delta(t)$ coincides. Let $q_j^{(k)}$ ($j = 1, 2, \ldots, N$; $k = 0, 1, \ldots, n$) represent $q_\Delta^{(k)}(t_j)$. Then on Γ_j

$$q_j^{(n-1}(t) = q_j^{(n-1)} \frac{t_j - t}{h_j} + q_j^{(n-1)} \frac{t - t_{j-1}}{h_j}, \tag{1}$$

where $h_j = t_j - t_{j-1}$. Define

$$\sigma_j = \frac{q_j^{(n-1)} - q_{j-1}^{(n-1)}}{h_j}, \qquad \tau_j = \sigma_{j+1} - \sigma_j . \tag{2}$$

We have from (2)

$$\sum_{j=1}^{N} \tau_j = 0 \tag{3}$$

and

$$\sum_{j=1}^{N} \sigma_j h_j = 0.$$

Inasmuch as

$$\sum_{j=1}^{N} \sigma_j h_j = \sum_{j=1}^{N} \sigma_j (t_j - t_{j-1}) = - \sum_{j=1}^{N} (\sigma_{j+1} - \sigma_j)\, t_j ,$$

we find that

$$\sum_{j=1}^{N} \tau_j t_j = 0. \tag{4}$$

It may be shown, moreover, that

$$\sum_{j=1}^{N} \tau_j t_j{}^k = 0 \qquad (k = 0, 1, 2, \ldots, n). \tag{5}$$

To see this, we write (1) in the form

$$q_j^{(n-1}(t) = q_{j-1}^{(n-1)} + \sigma_j (t - t_{j-1})$$

and integrate repeatedly. Then set $t = t_j$. We obtain

$$q_j^{(n-1)} - q_{j-1}^{(n-1)} = \sigma_j h_j$$

$$q_j^{(n-2)} - q_{j-1}^{(n-2)} = q_{j-1}^{(n-1)} h_j + \sigma \frac{h_j^2}{2!}$$

$$\cdots$$

$$q_j^{(k)} - q_{j-1}^{(k)} = q_{j-1}^{(k+1)} h_j + q_{j-1}^{(k+2)} \frac{h_j^2}{2!} + \cdots + q_{j-1}^{(n-1)} \frac{h_j^{n-k-1}}{(n-k-1)!} + \sigma_j \frac{h_j^{n-k}}{(n-k)!}. \qquad (6)$$

Multiply the left- and right-hand members of these equations by

$$\frac{(-1)^{n-k-1}}{(n-k)!} t_j^{n-k}, \qquad \frac{(-1)^{n-k-2}}{(n-k-1)!} t_j^{n-k-1}, \ldots, t_j,$$

respectively, add, and sum over j. Note that

$$\sum_j (q_j^{(m)} - q_{j-1}^{(m)}) t_j^{m-k+1} = -\sum q_{j-1}^{(m)} (t_j^{m-k+1} - t_{j-1}^{m-k+1}),$$

and that $t_{j-1} = t_j - h_j$. There results

$$\sum_{j=1}^{N} \Big\{ (-1)^{n-k} q_{j-1}^{(n-1)} \frac{t_j^{n-k} - (t_j - h_j)^{n-k}}{(n-k)!} + (-1)^{n-k-1} q_{j-1}^{(n-2)} \frac{t_j^{n-k-1} - (t_j - h_j)^{n-k-1}}{(n-k-1)!} + \cdots - q_{j-1}^{(k}[t_j - (t_j - h_j)] \Big\}$$
$$= \sum_{j=1}^{N} \Big\{ (-1)^{n-k+1} \frac{\sigma_j}{(n-k+1)!} \times [t_j^{n-k+1} - (t_j - h_j)^{n-k+1} + (-1)^{n-k+1} h_j^{n-k+1}] + (-1)^{n-k} \frac{q_{j-1}^{(n-1)}}{(n-k)!} [t_j^{n-k} - (t_j - h_j)^{n-k} + (-1)^{n-k} h_j^{n-k}] + \cdots - q_{j-1}^{(k)}[t_j - (t_j - h_j) - h_j] \Big\}.$$

Thus

$$\sum_{j=1}^{N} \Big\{ q_{j-1}^{(k)} h_j + q_{j-1}^{(k+1)} \frac{h_j^2}{2!} + \cdots + q_{j-1}^{(n-1)} \frac{h_j^{n-k}}{(n-k)!} + \sigma_j \frac{h_j^{n-k+1}}{(n-k+1)!} \Big\}$$
$$= -\sum_{j=1}^{N} \frac{(-1)^{n-k+1}}{(n-k)!} \sigma_j (t_j^{n-k+1} - t_{j-1}^{n-k+1}). \qquad (7)$$

The left-hand member of (7) is by (6) seen to be equal to

$$\sum_{j=1}^{N} [q_j^{(k-1)} - q_{j-1}^{(k-1)}] = 0.$$

By rearrangement of the right-hand side of (7), we obtain

$$\sum_{j=1}^{N} \tau_j t_j{}^k = 0, \qquad k = 0, 1,..., n. \tag{8}$$

We have derived the conditions (8) upon the assumption that a polynomial spline on Δ exists and that the quantities τ_j are jumps in the nth derivative at the mesh points, or knots, t_j ($j = 1, 2,..., N$). Conversely, if $\tau_1 ,..., \tau_N$ are any N quantities satisfying the $n + 1$ relations (8) ($N \geqslant n + 1$), then there exists a spline associated with the quantities τ_j, having $N - n - 1$ degrees of freedom remaining among the τ_j's. Thus we are led to the following theorem.

THEOREM 1. *A necessary and sufficient condition that $\tau_1 , \tau_2 ,..., \tau_N$ be the n-th derivative jumps of a spline of degree n ($N \geqslant n + 1$) on mesh Δ is that*

$$\sum_{j=1}^{N} \tau_j t_j{}^k = 0, \qquad k = 0, 1,..., n.$$

An obvious rearrangement of terms yields the corollary.

COROLLARY. *For arbitrary complex z, the necessary and sufficient conditions on τ_j may be replaced by*

$$\sum_{j=1}^{N} \tau_j (t_j - z)^k = 0 \qquad (k = 0, 1,..., n). \tag{9}$$

THE ANALYTIC SPLINE. Next we define in the region R the analytic spline $S_\Delta(z)$ associated with the complex polynomial spline $q_\Delta(t)$ on Γ. For this purpose we assume that Γ is rectifiable. Then for z in R,

$$S_\Delta(z) = \frac{1}{2\pi i} \int_\Gamma \frac{q_\Delta(t)\, dt}{t - z}. \tag{10}$$

We write, for t on Γ_j,

$$q_j(t) = q_{j-1} + q'_{j-1}(t - t_{j-1}) + \cdots + q_{j-1}^{(n-1)} \frac{(t - t_{j-1})^{n-1}}{n - 1)!} + \sigma_j \frac{(t - t_{j-1})^n}{n!}. \tag{11}$$

Consider the term

$$q_{j-1}^{(k)} \frac{(t - t_{j-1})^k}{k!} = q_{j-1}^{(k)} \frac{[(t - z) + (z - t_{j-1})]^k}{k},$$

which contributes to (10) the summand

$$\frac{1}{2\pi i} \frac{q_{j-1}^{(k)}}{k!} \int_{\Gamma_j} \Big\{ (t - z)^{k-1} + \binom{k}{1} (t - z)^{k-2}(z - t_{j-1})$$

$$+ \binom{k}{2} (t - z)^{k-3}(z - t_{j-1})^2 + \cdots$$

$$+ \binom{k}{k-1} (z - t_{j-1})^{k-1} + \frac{(z - t_{j-1})^k}{t - z} \Big\} \, dt$$

$$= \frac{1}{2\pi i} \frac{q_{j-1}^{(k)}}{k!} \Big\{ \frac{(t_j - z)^k - (t_{j-1} - z)^k}{k}$$

$$+ \binom{k}{1} \frac{(t_j - z)^{k-1} - (t_{j-1} - z)^{k-1}}{k - 1} (z - t_{j-1}) + \cdots$$

$$+ \binom{k}{k-1} \frac{(t_j - z) - (t_{j-1} - z)}{1} (z - t_{j-1})^{k-1}$$

$$+ (z - t_{j-1})^k \log \frac{t_j - z}{t_{j-1} - z} \Big\}$$

$$= \frac{1}{2\pi i} \frac{q_{j-1}^{(k)}}{k!} \Big\{ \frac{(t_j - z)^k - (t_{j-1} - z)^k}{k}$$

$$+ \frac{\binom{k}{1}}{k - 1} ([h_j - (t_j - z)](t_j - z)^{k-1} + (t_{j-1} - z)^k)$$

$$+ \frac{\binom{k}{2}}{k - 2} ([h_j - (t_j - z)]^2(t_j - z)^{k-2} - (t_{j-1} - z)^k) + \cdots$$

$$+ \frac{\binom{k}{k-1}}{1} ([h_j - (t_j - z)]^{k-1}(t_j - z) + (-1)^k(t_{j-1} - z)^k)$$

$$+ (z - t_{j-1})^k \log \frac{t_j - z}{t_{j-1} - z} \Big\}$$

$$= \frac{1}{2\pi i} \frac{q_{j-1}^{(k)}}{k!} [(t_j - z)^k - (t_{j-1} - z)^k]$$

$$\times \left(\frac{1}{k} - \frac{\binom{k}{1}}{k-1} + \frac{\binom{k}{2}}{k-2} - \cdots + (-1)^{k-1} \frac{\binom{k}{k-1}}{1} \right)$$

$$+ \frac{h_j^{k-1}}{(k-1)!} (t_j - z) + \frac{h_j^{k-2}}{k!} (t_j - z)^2$$

$$\times \left(\frac{\binom{k}{k-2}}{2} - \frac{\binom{k}{k-1}\binom{k-1}{1}}{1} \right) + \frac{h_j^{k-3}}{k!} (t_j - z)^3$$

$$\times \left(\frac{\binom{k}{k-3}}{3} - \frac{\binom{k}{k-2}\binom{k-1}{1}}{2} + \frac{\binom{k}{k-1}\binom{k-1}{2}}{1} \right) + \cdots$$

$$+ h_j(t_j - z)^{k-1} \left(\frac{\binom{k}{1}}{k-1} - \frac{\binom{k}{2}\binom{2}{1}}{k-2} + \frac{\binom{k}{3}\binom{3}{2}}{k-3} - \cdots \right.$$

$$\left. + (-1)^{k-2} \frac{\binom{k}{k-1}\binom{k-1}{k-2}}{1} \right) + (z - t_{j-1})^k \log \frac{t_j - z}{t_{j-1} - z}.$$

Set

$$E_k = \frac{1}{k!} \left[\frac{1}{k} \binom{k}{0} - \frac{1}{k-1} \binom{k}{1} + \frac{1}{k-2} \binom{k}{2} + \cdots + (-1)^{k-1} \binom{k}{k-1} \right].$$

Then the summand above is equal to

$$\frac{1}{2\pi i} q_{j-1}^{(k)} \Big\{ E_k[(t_j - z)^k - (t_{j-1} - z)^k] + E_1 \frac{h_j^{k-1}}{(k-1)!} (t_j - z)$$

$$+ E_2 \frac{h_j^{k-2}}{(k-2)!} (t_j - z)^2 + \cdots + E_{k-1} h_j (t_j - z)^{k-1}$$

$$+ \frac{1}{k!} (z - t_{j-1})^k \log \frac{t_j - z}{t_{j-1} - z}.$$

Similarly, $\sigma_j(t - t_{j-1})^n/n!$ contributes to (10) the summand

$$\frac{1}{2\pi i} \sigma_j \Big\{ E_n[(t_j - z)^n - (t_{j-k} - z)^n] + E_1 \frac{h_j^{n-1}}{(n-1)!} (t_j - z) + \cdots$$

$$+ E_{n-1} h_j (t_j - z)^{n-1} + \frac{1}{n!} (z - t_{j-1})^n \log \frac{t_j - z}{t_{j-1} - z} \Big\}.$$

Thus the integral in (10) may be written as

$$\frac{1}{2\pi i}\sum_{j=1}^{N}\left\{\sum_{k=0}^{n-1}\frac{q_{j-1}^{(k)}}{k!}(z-t_{j-1})^k+\frac{\sigma_j}{n!}(z-t_{j-1})^n\right\}\log\frac{t_j-z}{t_{j-1}-z}$$

$$+\frac{1}{2\pi i}\sum_{j=1}^{N}\left\{\sum_{k=1}^{n-1}q_{j-1}^{(k)}E_k[(t_j-z)^k-(t_{j-1}-z)^k]\right.$$

$$\left.+\sigma_j E_n[(t_j-z)^n-(t_{j-1}-z)^n]\right\} \tag{12}$$

$$+\frac{1}{2\pi i}\sum_{j=1}^{N}\left\{\sum_{k=2}^{n-1}q_{j-1}^{(k)}\sum_{p=1}^{k-1}E_p\frac{h_j^{k-p}}{(k-p)!}(t_j-z)^p\right.$$

$$\left.+\sigma_j\sum_{p=1}^{n-1}E_p\frac{h_j^{n-p}}{(n-p)!}(t_j-z)^p\right\}.$$

The first term may be written as

$$\frac{1}{2\pi i}\sum_{j=1}^{n}q_j(z)\log\frac{t_j-z}{t_{j-1}-z},$$

while the second term becomes, with changing the order of summation,

$$-\frac{1}{2\pi i}\sum_{k=1}^{n-1}E_k\sum_{j=1}^{N}[q_j^{(k)}-q_{j-1}^{(k)}](t_j-z)^k-\frac{1}{2\pi i}E_n\sum_{j=1}^{N}(\sigma_j-\sigma_{j-1})(t_j-z)^n,$$

which by (9) is equal to

$$-\frac{1}{2\pi i}\sum_{j=1}^{N}\sum_{k=1}^{n-1}E_k[q_j^{(k)}-q_{j-1}^{(k)}](t_j-z)^k$$

$$=-\frac{1}{2\pi i}\sum_{j=1}^{N}\sum_{k=1}^{n-1}E_k\left[q_{j-1}^{(k+1)}h_j+q_{j-1}^{(k+2)}\frac{h_j^2}{2!}+\cdots\right.$$

$$\left.+q_{j-1}^{(n-1)}\frac{h_j^{n-k-1}}{(n-k-1)!}+\sigma_j\frac{h_j^{n-k}}{(n-k)!}\right](t_j-z)^k$$

$$=-\frac{1}{2\pi i}\sum_{j=1}^{N}\sum_{k=1}^{n-1}E_k\left\{\sum_{p=k+1}^{n-1}q_{j-1}^{(p)}\frac{h_j^{p-k}}{(n-k)!}+\sigma_j\frac{h_j^{n-k}}{(n-k)!}\right\}(t_j-z)^k. \tag{13}$$

The third term in (12) is equal to

$$\frac{1}{2\pi i}\sum_{j=1}^{N}\left\{\sum_{k=2}^{n-1} q_{j-1}^{(k)}\sum_{p=1}^{k-1} E_p\frac{h_j^{k-p}}{(k-p)!}(t_j-z)^p+\sigma_j\sum_{p=1}^{n-1}E_p\frac{h_j^{n-p}}{(n-p)!}(t_j-z)^p\right\}$$

$$=\frac{1}{2\pi i}\sum_{j=1}^{N}\left\{\sum_{p=1}^{n-2}\sum_{k=p+1}^{n-1}E_p q_{j-1}^{(k)}\frac{h_j^{k-p}}{(k-p)!}(t_j-z)^p\right.$$

$$\left.+\sigma_j\sum_{p=1}^{n-1}E_p\frac{h_j^{n-p}}{(n-p)!}(t_j-z)^p\right\}.$$

Adding this to (13) gives zero. We have thus proved the following theorem

THEOREM 2. *Let Γ be a rectifiable Jordan curve, the mesh $\Delta : t_1, t_2, \ldots, t_N$ arranged on Γ in counterclockwise order, $q_\Delta(t)$ a polynomial spline on Γ of degree n, coincident on Γ_j with the polynomial $q_j(t)$. Then the corresponding analytic spline* (10) *is given by*

$$S_\Delta(z)=\frac{1}{2\pi i}\sum_{j=1}^{N} q_j(z)\log\frac{t_j-z}{t_{j-1}-z}. \tag{14}$$

An alternative form of (14) is of particular interest. Rearranging the sum we obtain

$$S_\Delta(z)=-\frac{1}{2\pi i}\sum_{j=1}^{N}[q_{j+1}(z)-q_j(z)]\log(t_j-z)+q_1(z). \tag{15}$$

It is necessary to specify the branch of the logarithm appearing in (12) and now in (15). In (12), we can choose the branch such that

$$\log\frac{t_j-z}{t_{j-1}-z}=\pi i, \tag{16}$$

when $z=(t_j+t_{j-1})/2$. Next choose a z_0 in R and a t_p such that t_p-z_0 is real and positive. Choose the branch of $\log(t_p-z_0)$ such that for z_0 in this position, $\log(t_p-z_0)$ is real. Now let t move along Γ counterclockwise from t_p to t_N, let $\log(t-z_0)$ be a continuous deformation of the branch of $\log(t_p-z_0)$ chosen, and require $\log(t_{p+1}-z_0),\ldots,\log(t_N-z_0)$ to coincide with $\log(t-z_0)$ when t is at $t_{p+1},\ldots,t_N$. Similarly specify the branches of $\log(t_{p-1}-z_0),\ldots,\log(t_1-z_0)$ by letting t move clockwise from t_p to t_1 along Γ. The branch of $\log(t_N-z_0)$ resulting from this choice gives

$$\log\frac{t_1-z_0}{t_N-z_0}=\pi i+2\pi i$$

when $z_0 = (t_1 + t_N)/2$. This is compensated for by the last term in (15). If we let z be arbitrary in R, we can continue $\log(t - z_0)$ to z along any Jordan arc γ in R. Since t is a point on Γ the continuation is independent of the particular choice of γ.

With reference to (11) we now write

$$q_{j+1}(z) = q_j + q_j'(z - t_j) + \cdots + q_j^{(n-1)} \frac{(z - t_j)^{n-1}}{(n-1)!} + \sigma_{j+1} \frac{(z - t_j)^n}{n!},$$

$$q_j(z) = q_j + q_j'(z - t_j) + \cdots + q_j^{(n-1)} \frac{(z - t_j)^{n-1}}{(n-1)!} + \sigma_j \frac{(z - t_j)^n}{n!},$$

whence

$$q_{j+1}(z) - q_j(z) = \tau_j \frac{(z - t_j)^n}{n!}. \tag{17}$$

Thus

$$S_\Delta(z) = \frac{(-1)^{n+1}}{2\pi i} \sum_{j=1}^{N} \tau_j \frac{(t_j - z)^n}{n!} \log(t_j - z) + q_1(z). \tag{18}$$

Further, in view of (9) we have

$$S_\Delta'(z) = \frac{(-1)^n}{2\pi i} \sum_{j=1}^{N} \tau_j \frac{(t_j - z)^{n-1}}{(n-1)!} \log(t_j - z) + q_1'(z).$$

$$\cdots$$

$$S_\Delta^{(n)}(z) = \frac{-1}{2\pi i} \sum_{j=1}^{N} \tau_j \log(t_j - z) + q_1^{(n)}(z), \tag{19}$$

$$S_\Delta^{(n+1)}(z) = \frac{1}{2\pi i} \sum_{j=1}^{N} \frac{\tau_j}{t_j - z},$$

$$\cdots$$

$$S_\Delta^{(k)}(z) = \frac{(k-n-1)!}{2\pi i} \sum_{j=1}^{N} \frac{\tau_j}{(t_j - z)^{k-n}}, \quad k \geqslant n + 1.$$

The equation associated with $k = n + 1$ in (19) bears a direct relationship to the Runge theorem on approximation by rational functions with prescribed poles on Γ. This has been explored in the case of the complex cubic spline [1].

In what follows we shall refer to $S_\Delta(z)$, which is associated with a polynomial spline of degree n on Γ, as an *analytic spline of degree n* interior to π.

Multiple Interpolation at an Interior Point

Given a rectifiable Jordan curve Γ, a mesh $\Delta : t_1, t_2, \ldots, t_N$ on Γ and a point z_0 interior to Γ, we shall determine the associated analytic spline $S_\Delta(z)$ of degree n ($n \leqslant N-1$) such that $S^{(k)}(z_0)$ ($k = 0, 1, \ldots, N-1$) takes on the prescribed value A_k. We solve for the N quantities τ_j using the $n+1$ equations (9) together with the $N-n-1$ equations obtained from (19):

$$\frac{1}{2\pi i}\sum_{j=1}^{N}\frac{\tau_j}{(t_j-z_0)^k} = \frac{A_{n+k}}{(k-1)!} \qquad (k = 1, 2, \ldots, N-n-1).$$

The resulting system is written

$$\begin{bmatrix} 1 & 1 & & 1 \\ (t_1-z_0)^{-1} & (t_2-z_0)^{-1} & \cdots & (t_N-z_0)^{-1} \\ \vdots & \vdots & & \vdots \\ (t_1-z_0)^{-n} & (t_2-z_0)^{-n} & \cdots & (t_N-z_0)^{-n} \\ (t_1-z_0)^{-n-1} & (t_2-z_0)^{-n-1} & \cdots & (t_N-z_0)^{-n-1} \\ \vdots & \vdots & & \vdots \\ (t_1-z_0)^{-N+1} & (t_2-z_0)^{-N+1} & \cdots & (t_N-z_0)^{-N+1} \end{bmatrix} \begin{bmatrix} \tau_1(t_1-z_0)^n \\ \tau_2(t_2-z_0)^n \\ \vdots \\ \tau_{n+1}(t_{n+1}-z_0)^n \\ \tau_{n+2}(t_{n+2}-z_0{}^n) \\ \vdots \\ \tau_N(t_N-z_0)^n \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 2\pi i\,\dfrac{A_{n+1}}{0!} \\ \vdots \\ 2\pi i\,\dfrac{A_{N-1}}{(N-n-2)!} \end{bmatrix}, \qquad (20)$$

where the relationship of the coefficient matrix to the Vandermonde matrix is evident. Define

$$s_j(z_0) = \prod_{\substack{k=1 \\ k\neq j}}^{N} (z_0 - t_k). \qquad (21)$$

Then, differentiating with respect to z_0 gives

$$s_j'(z_0) = [\text{sum of products of } (z_0 - t_k),\ k \neq j, \text{ taken } N-2 \text{ at a time}],$$

$$\frac{1}{2!}\, s_j''(z_0) = [\text{sum of products of } (z_0 - t_k),\ k \neq j, \text{ taken } N-3 \text{ at a time}],$$

$$\frac{1}{(N-2)!}\,s_j^{(N-2)}(z_0) = [\text{sum of products of } (z_0 - t_k),\ k \neq j, \text{ taken 1 at a time}],$$

$$\frac{1}{(N-1)!}\,s_j^{(N-1)}(z_0) = 1.$$

The inverse of the coefficient matrix in (20) is

$$\begin{bmatrix} (t_1 - z_0)^{N-1}/s_1(t_1) & (t_1 - z_0)^{N-1}\,s_1^{(N-2)}(z_0)/(N-2)!\,s_1(t_1) & \cdots & (t_1 - z_0)^{N-1}\,s_1(z_0)/s_1(t_1) \\ (t_2 - z_0)^{N-1}/s_2(t_2) & (t_2 - z_0)^{N-1}\,s_2^{(N-2)}(z_0)/(N-2)!\,s_2(t_2) & \cdots & (t_2 - z_0)^{N-1}\,s_2(z_0)/s_2(t_2) \\ \vdots & \vdots & & \vdots \\ (t_N - z_0)^{N-1}/s_N(t_N) & (t_N - z_0)^{N-1}\,s_N^{(N-2)}(z_0)/(N-2)!\,s_N(t_N) & \cdots & (t_N - z_0)^{N-1}\,s_N(z_0)/s_N(t_N) \end{bmatrix}, \tag{22}$$

for if we multiply (22) on the right by the coefficient matrix in (20), we obtain for the element in the jth row and kth column

$$\begin{aligned} &\frac{(t_j - z_0)^{N-1}}{s_j(t_j)}\left\{1 + \frac{s_j^{(N-2)}(z_0)}{(N-2)!}(t_k - z_0)^{-1} + \cdots + \frac{s_j(z_0)}{1!}(t_k - z_0)^{-N+1}\right\} \\ &= \left(\frac{t_j - z_0}{t_k - z_0}\right)^{N-1}\left\{s_j(z_0) + s_j'(z_0)(t_k - z_0) + \cdots\right. \\ &\qquad \left. + \frac{s_j^{(N-2)}(z_0)}{(N-2)!}(t_k - z_0)^{N-1} + \frac{s_j^{(N-1)}(z_0)}{(N-1)!}(t_k - z_0)^{N-1}\right\}\frac{1}{s_j(t_j)} \\ &= \left(\frac{t_j - z_0}{t_k - z_0}\right)^{N-1}\frac{s_j(t_k)}{s_j(t_j)} = \delta_{jk}\,. \end{aligned}$$

It follows, if $N \geqslant n + 2$, that for $j - 1, 2, \ldots, N$ we have

$$\begin{aligned} \tau_j(t_j - z_0)^n = \frac{2\pi i(t_j - z_0)^{N-1}}{s_j(t_j)}\,\frac{A_{n+1}}{0!}&\left\{\frac{s_j^{(N-n-2)}(z_0)}{(N-n-2)!} + \cdots\right. \\ &\left. + \frac{A_{N-1}}{(N-n-2)!}\,\frac{s_j(z_0)}{0!}\right\}. \end{aligned} \tag{23}$$

If $N = n + 1$, $\tau_1 = \tau_2 = \cdots = \tau_N = 0$. In order to complete the definition of $S_\Delta(z)$, it is necessary to determine the polynomial $q_1(z)$. If $N = n + 1$ (and also if $N < n + 1$ and $A_{n+1} = A_{n+2} = \cdots = A_{N-1} = 0$, since here again $\tau_1 = \tau_2 = \cdots = \tau_N = 0$), we have

$$q_1(z) \equiv A_0 + A_1(z - z_0) + \cdots + \frac{A_n}{n!}(z - z_0)^n \equiv S_\Delta(z).$$

If $N > n + 1$, then the quantities $q_1^{(k)}(z_0)$ $(k = 0, 1, 2,..., n)$ are determined by (18) and the first n equations of (19) by setting $z = z_0$:

$$A_k = S_\Delta^{(k)}(z_0) = \frac{(-1)^{n-k+1}}{2\pi i} \sum_{j=1}^{N} \tau_j(t_j - z)^{n-k} \log(t_j - z_0) + q_1^{(k)}(z_0). \tag{24}$$

Of course, we may also have the formula

$$S_\Delta(z) = \sum_{p=0}^{n} \frac{1}{p!} A_p(z - z_0)^p + \sum_{p=n+1}^{\infty} \frac{1}{p!} (z - z_0)^p \frac{(p-n-1)!}{2\pi i} \sum_{j=1}^{N} \frac{\tau_j}{(t_j - z_0)^{p-n}}, \tag{25}$$

as an alternative to (18). We have then the following theorem:

THEOREM 3. *Let Γ be a rectifiable Jordan curve and $\Delta : t_1, t_2,..., t_N$ be a mesh on Γ. Let $q_\Delta(z)$ be a complex spline of degree n on Γ $(n \leqslant N - 1)$ associated with mesh Δ, and let $S_\Delta(z)$ be the corresponding analytic spline defined within Γ. If the quantities A_k $(k = 0, 1,..., N - 1)$ are prescribed and z_0 is a point interior to C, then the analytic spline $S_\Delta(z)$ for which*

$$S_\Delta^{(k)}(z_0) = A_k \qquad (k = 0, 1,..., N - 1)$$

exists and is unique.

We note that if the A_k $(k = 0, 1,..., N - 1)$ are the values of the derivatives of a function $f(z)$ at z_0, $A_k = f^{(k)}(z_0)$, then (23) may be written as

$$\tau_j = \frac{2\pi i}{s_j(t_j)} \frac{(t_j - z_0)^{N-n-1}}{(N - n - 2)!} \frac{d^{N-n-2}}{(dz_0)^{N-n-2}} [f^{(n+1)}(z_0)\, s_j(z_0)]. \tag{26}$$

If $N < n + 1$, the Eqs. (20) lead to $\tau_1 = \tau_2 = \cdots = \tau_N = 0$. In this case the spline exists, is a polynomial with N constraints on its $n + 1$ coefficients, and so is not unique.

UNIFORM MESH ON THE UNIT CIRCLE. For the special case in which Γ is the unit circle, $z_0 = 0$, and $t_j = \omega^j$, where $\omega = e^{2\pi i/N}$, we have

$$s_j(z_0) = \frac{{z_0}^N - 1}{z_0 - \omega^j}, \qquad z_0 \neq \omega^j, \tag{27}$$

so that

$$\begin{aligned} s_j(0) &= t_j^{N-1}, \\ s_j'(0) &= t_j^{N-2}, \\ \cdots \\ \frac{s_j^{(N-n-2)}(0)}{(N-n-2)!} &= t_j^{n+1}, \end{aligned} \tag{28}$$

and

$$s_j(t_j) = \lim_{t \to t_j} \frac{t^{N-1}}{t - t_j} = N t_j^{N-1}. \tag{29}$$

When $A_k = f^{(k)}(0)$, Eq. (23) becomes

$$\tau_j = \frac{2\pi i t_j^{N-n-1}}{N t_j^{N-1}} \left\{ \frac{f^{(n+1)}(0)}{0!} t_j^{n+1} + \frac{f^{(n+2)}(0)}{1!} t_j^{n+2} + \cdots + \frac{f^{(N-1)}(0)}{(N-n-2)!} t_j^{N-1} \right\} = \frac{2\pi i}{N} t_j p^{(n+1)}(t_j), \tag{30}$$

where

$$p(z) = f(0) + f'(0)\, z + \cdots + \frac{f^{(N-1)}(z)}{(N-1)!} z^{N-1}. \tag{31}$$

Assume now that $f(z)$ is analytic at $z = 0$. Require $N \geqslant n + 1$. We have already indicated that if $f(z)$ is a polynomial of degree $\leqslant n$, then $S_\Delta(z) \equiv q_1(z) \equiv f(z)$.

In general, if we expand $f(z)$ in a Taylor series about $z = 0$, so that with $A_k = f^{(k)}(0)$ we have

$$f(z) = \sum_{p=0}^{\infty} A_p z^p,$$

then by (25) and (30) we have

$$\begin{aligned} S_\Delta(z) = {} & \sum_{p=0}^{n} \frac{1}{p!} A_p z^p + \sum_{p=n+1}^{\infty} \frac{1}{p!} z^p \frac{(p-n-1)!}{2\pi i} \\ & \cdot \sum_{j=1}^{N} \frac{2\pi i}{N} \frac{t_j^{N-n-1}}{t_j^{p-n} t_j^{N-1}} \left\{ \frac{A_{n+1}}{0!} t_j^{n+1} \right. \\ & \left. + \frac{A_{n+2}}{1!} t_j^{n+2} \cdots + \frac{A_{N+1}}{(N-n-2)!} t_j^{N-1} \right\} = \sum_{p=0}^{n} \frac{1}{p!} A_p z^p \\ & + \sum_{p=n+1}^{N-1} \frac{A_p}{(p-n-1)!} \left[\frac{(p-n-1)!}{p!} z^p + \frac{(N+p-n-1)!}{(N+p)!} z^{N+p} \right. \\ & \left. + \frac{(2N+p-n-1)!}{(2N+p)!} z^{2N+p} + \cdots \right]. \end{aligned} \tag{32}$$

Differentiation $(n+1)$ times and noting that $\sum_{j=1}^{N} t_j^{\,k} = N$ for k a multiple of N, zero otherwise, gives

$$S_\Delta^{(n+1)}(z) = \sum_{p=n+1}^{N-1} \frac{A_p}{(p-n-1)!} \frac{z^{p-n-1}}{1-z^N} = \sum_{p=n+1}^{N-1} \frac{f^{(p)}(0)}{(p-n-1)!} \frac{z^{p-n-1}}{1-z^N}, \tag{33}$$

and repeated integration then yields

$$S_\Delta(z) = \sum_{p=0}^{n} \frac{f^{(p)}(0)}{p!} z^p + \frac{1}{n!} \sum_{p=n+1}^{N-1} \frac{f^{(p)}(0)}{(p-n-1)!} \int_0^z \frac{(z-t)^n t^{p-n-1}\, dt}{1-t^N}. \tag{34}$$

For purposes of reference we also exhibit the polynomial $q_1(z)$. We have from (19), for $k = 0, 1, \ldots, n$, the relations

$$\begin{aligned} f^{(k)}(0) = S_\Delta^{(k)}(0) = {} & \frac{(-1)^{n+1-k}}{2\pi i} \sum_{j=1}^{N} \frac{t_j^{n-k}}{(n-k)!} \frac{2\pi i j}{N} \cdot \frac{2\pi i}{N} \\ & \cdot \left\{ \frac{f^{(n+1)}(0)}{0!} t_j + \frac{f^{(n+2)}(0)}{1!} t_j^2 + \cdots \right. \\ & \left. + \frac{f^{(N-1)}(0)}{(N-n-2)!} t_j^{N-n-1} \right\} + q_1^{(k)}(0). \end{aligned}$$

Since $\sum_{j=1}^{N} j t_j^{\,k} = \sum_{j=1}^{N} j\omega^{kj} = N\omega^k/(\omega^k - 1)$ for $1 \leqslant k \leqslant N-1$, this gives

$$\begin{aligned} q_1^{(k)}(0) = f^{(k)}(0) + {} & \frac{(-1)^{n-k}}{(n-k)!} \frac{2\pi i}{N} \left\{ \frac{f^{(n+1)}(0)}{0!} \frac{\omega^{n-k+1}}{\omega^{n-k+1}-1} \right. \\ & \left. + \frac{f^{(n+2)}(0)}{1!} \frac{\omega^{n-k+2}}{\omega^{n-k+2}-1} + \cdots + \frac{f^{(N-1)}(0)}{(N-n-2)!} \frac{\omega^{N-n-1}}{\omega^{N-k-1}-1} \right\} \end{aligned}$$

if $N > n+1$; if $N = n+1$, the first term in braces is replaced for $k = 0$ by $f^{(n+1)}(0)\ (N+1)/2$ while other terms remain unchanged. Thus

$$\begin{aligned} q_1(z) = {} & \sum_{k=0}^{n} \frac{1}{k!} q_1^{(k)}(0)\, z^k = \sum_{k=0}^{n} f^{(k)}(0) \frac{z^k}{k!} \\ & + \frac{2\pi i}{N} \sum_{p=n+1}^{N-1} \frac{f^{(p)}(0)}{(p-n-1)!} \sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} \frac{\omega^{p-k}}{\omega^{p-k}-1} \frac{z^k}{k!}. \end{aligned}$$

Let now $f(z)$ be analytic in $|z| < r \leqslant 1$ but not throughout $|z| < r^*$ for any $r^* > r$. We may write in $|z| < r$

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad \limsup_{k\to\infty} |a_k|^{1/k} = 1/r.$$

In $|z| < r$, set

$$R_{p+1}(z) = f(z) - \sum_{k=0}^{p} a_k z^k.$$

Then by (33)

$$S_{\Delta}^{(n+1)}(z) = \sum_{p=n+1}^{N-1} \frac{p!\, a_p}{(p-n-1)!} \frac{z^{p-n-1}}{1-z^N}$$

$$= \frac{1}{1-z^N} [f^{(n+1)}(z) - R_N^{(n+1)}(z)] \tag{35}$$

and

$$S_{\Delta}(z) = \sum_{k=0}^{n} a_k z^k + \frac{1}{n!} \int_0^z \frac{(z-t)^n [f^{(n+1)}(t) - R_N^{(n+1)}(t)]\, dt}{1-t^N}. \tag{36}$$

Convergence Properties in the Unit Circle. It is possible now to present convergence properties of these analytic splines to $f(z)$. Let Δ_k be a mesh of N_k equispaced points on

$$|z| = 1, \qquad \Delta_k = \{t_{k,1}, t_{k,2}, \ldots, t_{k,N_k}\}, \qquad t_{k,j} = \exp[2\pi ij/N_k]$$

and $N_{k'} > N_k$ when $k' > k$. The analytic spline $S_{\Delta_k}(z)$ has the origin as an N_k-fold point of interpolation to $f(z)$. For $|z| < r$,

$$f^{(n+1)}(z) - S_{\Delta_k}^{(n+1)}(z) = \frac{-z^{N_k}}{1-z^{N_k}} f^{(n+1)}(z) + \frac{1}{1-z^{N_k}} R_{N_k}^{(n+1)}(z).$$

For $|z| \leqslant r' < r$, $|f^{(n+1)}(z)|$ has a bound

$$|f^{(n+1)}(z)| \leqslant M(r'),$$

and we have on $|z| \leqslant r'$ for arbitrary ϵ, $0 < \epsilon < r$, and all N_k sufficiently large

$$|R_{N_k}^{(n+1)}(z)| \leqslant \frac{\left(\dfrac{r'}{r-\epsilon}\right)^{N_k-n-1}}{1-\dfrac{r'}{r-\epsilon}} \cdot \frac{N_k!}{(N_k-n-1)!}.$$

Now by Stirling's formula

$$\frac{N_k!}{(N_k-n-1)!} \leqslant N_k^{n+1}(1+0(1)),$$

and so for arbitrary b, $r'/(r-\epsilon) < b < 1$, we have

$$\frac{N_k!}{(N_k-n-1)!} \frac{r-\epsilon}{r-r'-\epsilon} \left(\frac{r'}{r-\epsilon}\right)^{N_k-n-1} < b^{N_k}$$

when N_k is sufficiently large. Thus for $|z| \leqslant r'$, arbitrary $\epsilon < r - r'$, arbitrary b with $r'/(r-\epsilon) < b < 1$, and N_k sufficiently large we have

$$|f^{(n+1)} - S_{\Delta_k}^{(n+1)}(z)| \leqslant \frac{1}{1-(r')^{N_k}} [(r')^{N_k} M(r') + b^{N_k}].$$

From the arbitrary character of ϵ and b we now obtain

$$\limsup_{N_k\to\infty} [\max |f^{(n+1)}(z) - S_{\Delta_k}^{(n+1)}(z)|, |z| \leqslant r']^{1/N_k} \leqslant r'/r.$$

The inequality is eliminated since the degree of convergence r'/r is optimal [5].

The companion relation on the sequence $\{S_{\Delta k}(z)\}$ itself is an immediate consequence of (35): for $r \leqslant r' < 1$ we see that

$$\limsup_{N_k\to\infty} [\max | S_{\Delta_k}^{(n+1)}(z)|, |z| \leqslant r']^{1/N_k} \leqslant r'/r,$$

and this is again optimal. Thus we have proved

THEOREM 4. *Let $f(z)$ be analytic in $|z| < r \leqslant 1$. Let $\{\Delta_k\}$ be a sequence of meshes on $|z| = 1$ with $\Delta_k = \{t_{k,1}, t_{k,2},..., t_{k,N_k}\}$, $t_{k,j} = \exp[2\pi ij/N_k]$, and $N_{k'} > N_k$ when $k' > k$. Then for $r' < r$,*

$$\limsup_{N_k\to\infty} [\max |f^{(n+1)}(z) - S_{\Delta_k}^{(n+1)}(z)|, |z| = r']^{1/N_k} = r'/r. \tag{37}$$

For $r \leqslant r' < 1$,

$$\limsup_{N_k\to\infty} [\max | S_{\Delta_k}^{(n+1)}(z)|, |z| \leqslant r']^{1/N_k} = r'/r. \tag{38}$$

Integration $n+1$ times and using the interpolation property,

$$S_k^{(p)}(0) = f^{(p)}(0), \qquad p = 0, 1,..., n,$$

gives

THEOREM 5. *Under the conditions of Theorem 4, for $r' < r$ and arbitrary positive $\epsilon < r - r'$, we have*

$$[\max |f^{(p)}(z) - S_{\Delta_k}^{(p)}(z)|, |z| \leqslant r'] \leqslant \text{const.} \left(\frac{r'}{r-\epsilon}\right)^{N_k} (r')^{n+1-p}, \tag{39}$$

and for $r < r' < 1$ and arbitrary $\epsilon > 0$

$$[\max | S_{\Delta_k}^{(p)}(z)|, |z| \leqslant r'] \leqslant \text{const.} \left(\frac{r'+\epsilon}{r}\right)^{N_k} \cdot r'^{n+1-p}. \tag{40}$$

The above results are characteristic of approximation by varies types of extremal polynomials and rational functions.

CONCLUDING REMARKS. In our earlier paper [1] on complex cubic splines, we explored problems of convergence, both of complex cubic splines interpolating on the boundary of a region and of associated analytic cubic splines in the closed interior of the region. In the present paper the convergence results on polynomial splines have been restricted to those which are associated with uniform meshes on the unit circle and multiple interpolation at the origin. In forthcoming papers we study in detail the relations between these two types of interpolating polynomial splines, for the unit circle and for more general regions.

It is also now clear, from the proof of Theorem 5, that like many extremal types of approximating functions, analytic splines of multiple interpolation at a point are incisive enough in their approximation to permit determination of structural properties of the analytic function being approximated. This intimate relationship is explored in detail in subsequent papers.

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